Introduction to Analytical Target Cascading

Michael Kokkolaras

ME 555 Guest Lecture

Winter 2004

Outline

- Analytical Target Cascading (ATC): When, Why, What
- Mathematical Formulation
  - Bi-level, one subsystem only
  - Bi-level, two subsystems, without shared variables
  - Bi-level, two subsystems, with shared variables
  - Bi-level, general

General Nonlinear Optimization Problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x},\mathbf{p}) \\ \text{with respect to} \quad & \mathbf{x} \\ \text{subject to} \quad & \mathbf{g}(\mathbf{x},\mathbf{p}) \le \mathbf{0} \\ & \mathbf{h}(\mathbf{x},\mathbf{p}) = \mathbf{0} \end{aligned}$$

where **x** are design variables, **p** are parameters, f is a scalar function, and **g** and **h** are vector functions; and where at least one of the above functions is nonlinear

Optimal Design Laboratory

Design vs. Analysis Models

- The problem formulated in the previous slide is referred to as "design model"; it consists of an objective function, design variables, parameters, and equality/inequality constraints (the latter are not required; if there aren't any, we simply have an unconstrained optimization problem)
- Analysis models are used to compute the function values of the objective and the constraints. These can be analytic expressions, spreadsheets, or simulations

Important Distinctions

- The analysis or simulation models used in design optimization have many inputs; we need to identify which ones are both interesting and "controllable"; we treat those as design variables; the rest are parameters
  - Design variables are manipulated (controlled) by the optimization algorithm
  - Parameters are held fixed during the optimization process (e.g., Young's modulus of a material); parametric studies are performed to assess their influence on the optimization results
- Constants are fixed "universally" (e.g., π)

Design Target Matching Problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{R}(\mathbf{x}), \mathbf{T}) \\ \text{with respect to} \quad & \mathbf{x} \\ \text{subject to} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{aligned}$$

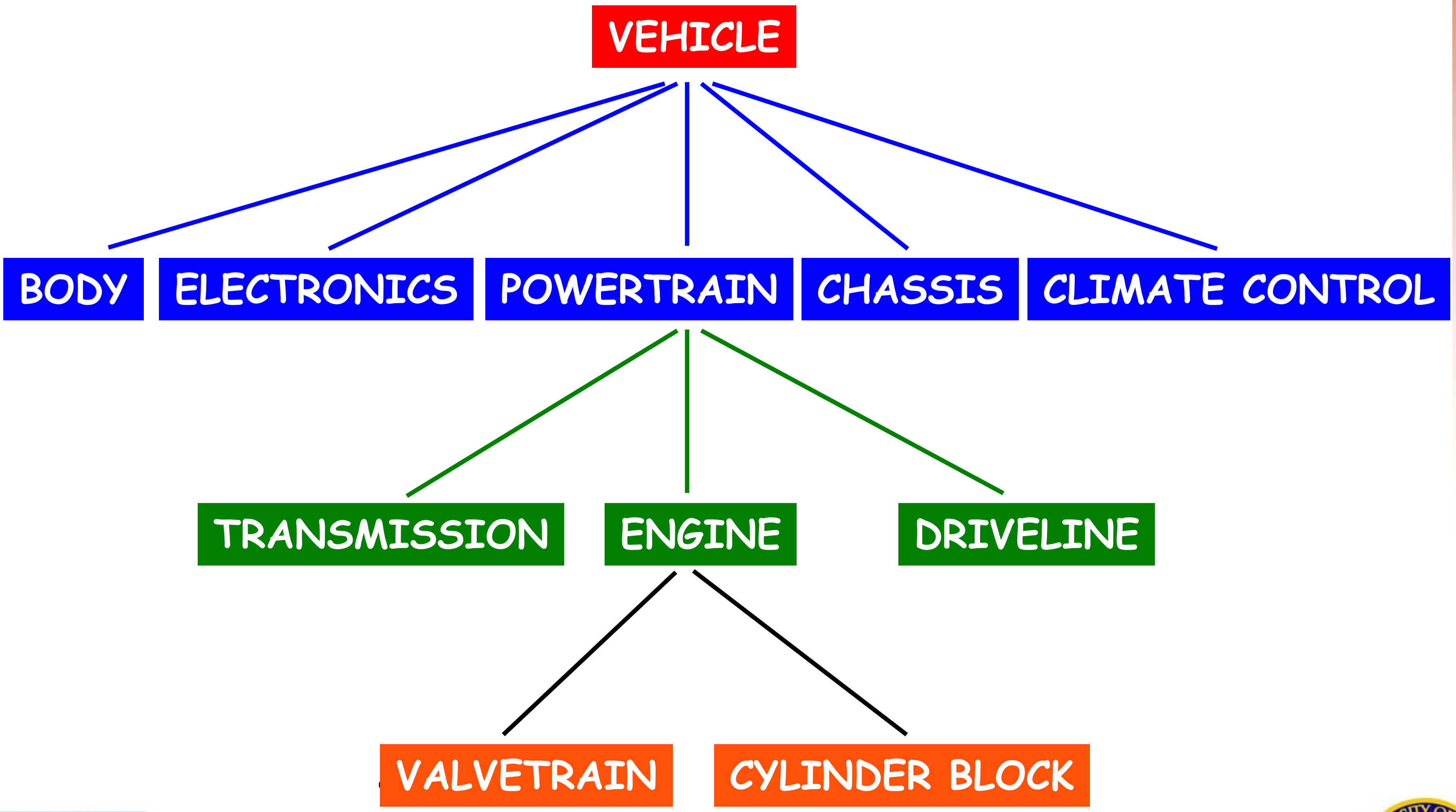
where $\mathbf{R}(\mathbf{x})$ are system responses and $\mathbf{T}$ are design targets (parameters);

f is a function that measures deviations of responses from design targets

Design by Decomposition

- When dealing with the optimal design of engineering systems, the design target matching problem is often impossible to solve as formulated due to its large size and complexity
- Original problem is decomposed into a set of linked subproblems
- Typically, the partitioning reflects the structure of the organization so that different design teams are assigned with different subproblems according to expertise

Decomposition Example



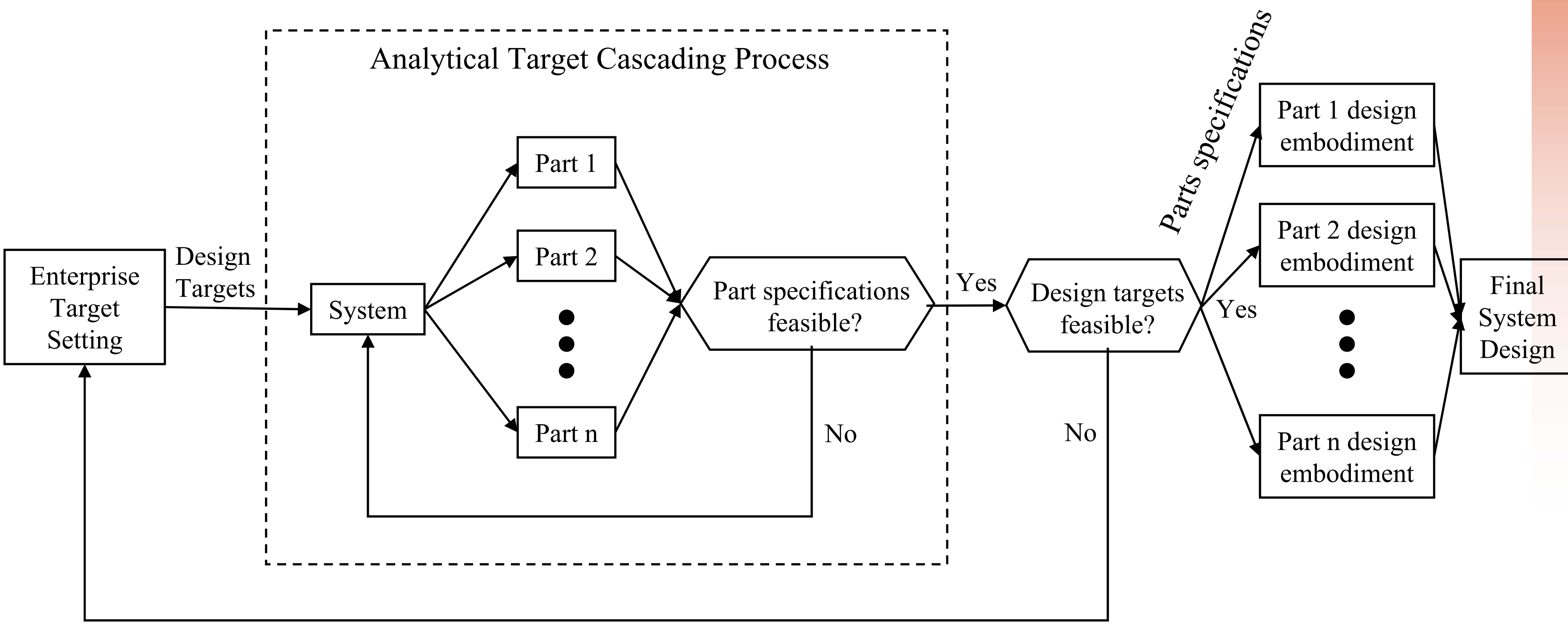
Challenges

- Need to assign design targets for the subproblems to the design teams
- Design teams may focus on own goals without taking into consideration interactions with other subproblems; this will compromise design consistency and optimality of the original problem

Analytical Target Cascading (ATC)

- Is a mathematical methodology to translate ("cascade") system design targets to subsystem, component, part, etc. design specifications in a consistent and efficient manner
- Consistent means that subsystems, components, parts, etc. are designed such that possible subsystem, component, part, etc. design objective conflicts do not compromise system design
- Efficient means that ATC is employed in the early stages of product development process, avoiding late (and costly) design iterations
- ATC enables concurrent engineering and outsourcing; once design specifications have been determined, detailed subsystem design embodiment can be performed in parallel

ATC in Product Development



Assumptions

- The system always resides at the top level, and is decomposed into subsystems, the subsystems are decomposed into components, the components into parts, and so on
- The decomposition results into a multilevel hierarchy consisting of "elements" that are associated with the subproblems
- At least one output of a "child"-element has to be an input to its "parent"-element
- Children of the same parent may share one or more design variables
- Decompositions do not need to be "complete"

Definitions

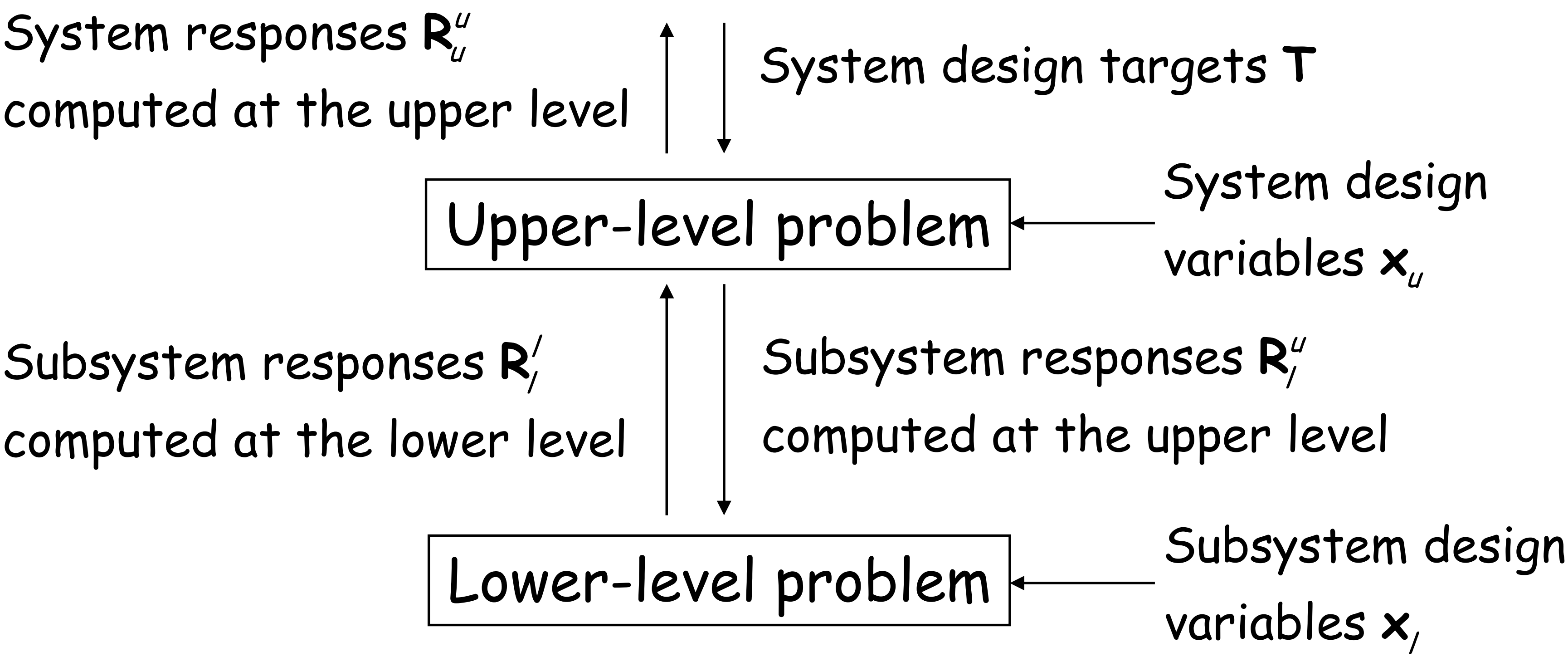
- Parent responses $\mathbf{R}_p$ are functions of
  - Children response variables $\mathbf{R}_{c1}$, $\mathbf{R}_{c2}$, …, $\mathbf{R}_{cn}$, (required)
  - Local design variables $\mathbf{x}_p$ (optional)
  - Shared design variables $\mathbf{y}_p$ (optional)

$$\mathbf{R}_p = \mathbf{r}(\mathbf{R}_{c_1}, \ldots, \mathbf{R}_{c_n}, \mathbf{x}_p, \mathbf{y}_p)$$

- In the following formulations:
  - Subscript index pairs denote level and element
  - Superscript indices denote association of computation (location where values are obtained at); necessary for response and shared design variable types only
  - Special cases may allow simplifying notation

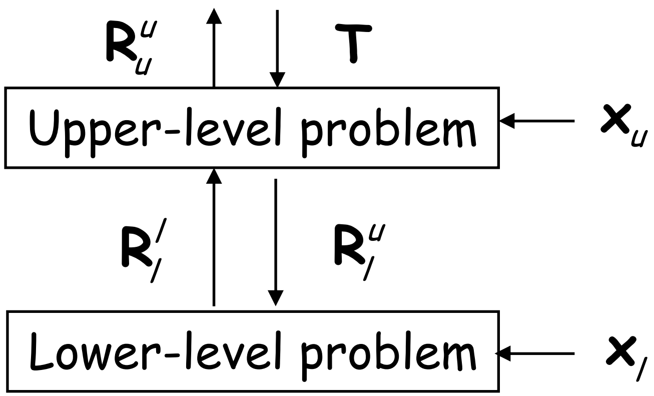
Bi-level hierarchy, one subsystem only

Information Flow



Bi-level hierarchy, one subsystem only

Upper-level problem formulation, first ATC iteration



$$\text{minimize} \quad \left\| \mathbf{R}_u^u - \mathbf{T} \right\|_2^2$$

$$\text{with respect to} \quad \mathbf{R}_l^u, \mathbf{x}_u$$

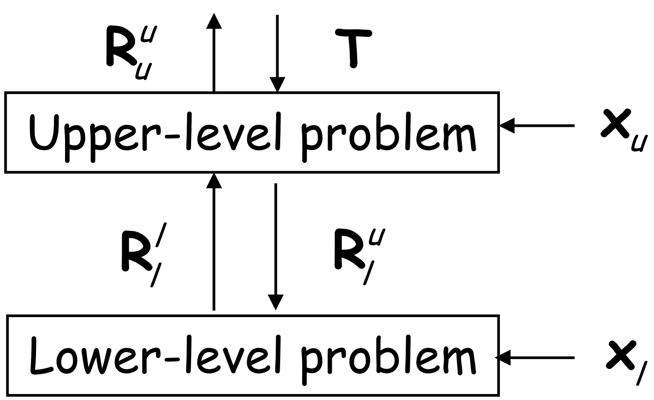
$$\text{subject to} \quad \mathbf{g}_u(\mathbf{R}_l^u, \mathbf{x}_u) \leq \mathbf{0}$$

$$\mathbf{h}_u(\mathbf{R}_l^u, \mathbf{x}_u) = \mathbf{0}$$

$$\text{where} \quad \mathbf{R}_u^u = \mathbf{r}_u(\mathbf{R}_l^u, \mathbf{x}_u)$$

Bi-level hierarchy, one subsystem only

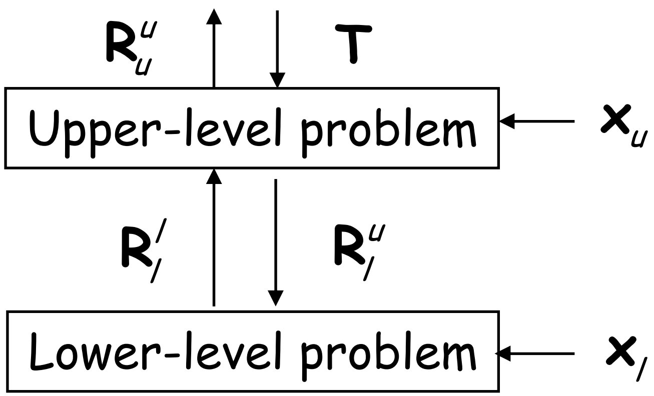
Lower-level problem formulation, ATC iterations 1,2,…,n



$$
\begin{aligned}
&\text{minimize} && \left\| \mathbf{R}_l^l - \mathbf{R}_l^u \right\|_2^2 \\
&\text{with respect to} && \mathbf{x}_l \\
&\text{subject to} && \mathbf{g}_l(\mathbf{x}_l) \le 0 \\
& && \mathbf{h}_l(\mathbf{x}_l) = 0 \\
&\text{where} && \mathbf{R}_l^l = \mathbf{r}_l(\mathbf{x}_l)
\end{aligned}
$$

Bi-level hierarchy, one subsystem only

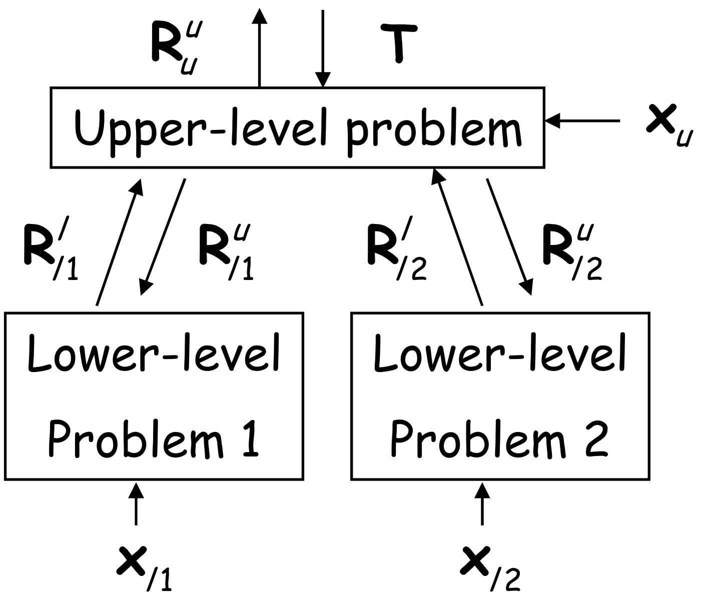
Upper-level problem formulation, ATC iterations 2,3,…,n



$$
\begin{aligned}
\text{minimize} \quad & \left\| \mathbf{R}_u^u - \mathbf{T} \right\|_2^2 + \varepsilon_R^u \\
\text{with respect to} \quad & \mathbf{R}_l^u, \mathbf{x}_u, \varepsilon_R^u \\
\text{subject to} \quad & \left\| \mathbf{R}_l^u - \mathbf{R}_l^l \right\|_2^2 \le \varepsilon_R^u \\
& \mathbf{g}_u(\mathbf{R}_l^u, \mathbf{x}_u) \le \mathbf{0} \\
& \mathbf{h}_u(\mathbf{R}_l^u, \mathbf{x}_u) = \mathbf{0} \\
\text{where} \quad & \mathbf{R}_u^u = \mathbf{r}_u(\mathbf{R}_l^u, \mathbf{x}_u)
\end{aligned}
$$

Two subsystems, no shared variables

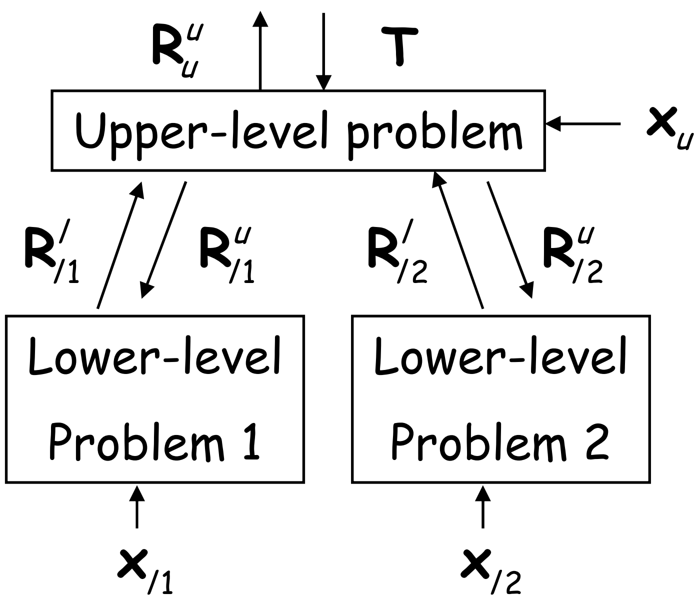
Upper-level problem formulation, first ATC iteration



$$
\begin{aligned}
&\text{minimize} && \left\| \mathbf{R}_u^u - \mathbf{T} \right\|_2^2 \\
&\text{with respect to} && \mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u \\
&\text{subject to} && \mathbf{g}_u(\mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u) \leq \mathbf{0} \\
& && \mathbf{h}_u(\mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u) = \mathbf{0} \\
&\text{where} && \mathbf{R}_u^u = \mathbf{r}_u(\mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u)
\end{aligned}
$$

Two subsystems, no shared variables

Lower-level problem formulation, ATC iterations 1,2,…,n

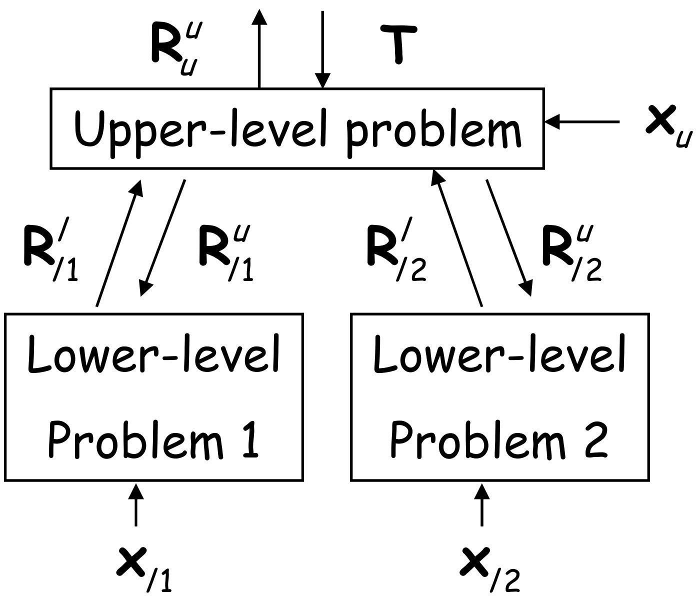


minimize	$\left\Vert \mathbf{R}_{lk}^l - \mathbf{R}_{lk}^u \right\Vert_2^2$
with respect to	$\mathbf{x}_{lk}$
subject to	$\mathbf{g}_{lk}(\mathbf{x}_{lk}) \leq \mathbf{0}$
	$\mathbf{h}_{lk}(\mathbf{x}_{lk}) = \mathbf{0}$
where	$\mathbf{R}_{lk}^l = \mathbf{r}_{lk}(\mathbf{x}_{lk})$

for k=1,2

Two subsystems, no shared variables

Upper-level problem formulation, ATC iterations 2,3,…,n



minimize $\left\|\mathbf{R}_u^u - \mathbf{T}\right\|_2^2 + \varepsilon_R^u$

with respect to $\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \mathbf{x}_u, \varepsilon_R^u$

subject to $\left\|\mathbf{R}_{l1}^u - \mathbf{R}_{l1}^l\right\|_2^2 + \left\|\mathbf{R}_{l2}^u - \mathbf{R}_{l2}^l\right\|_2^2 \le \varepsilon_R^u$

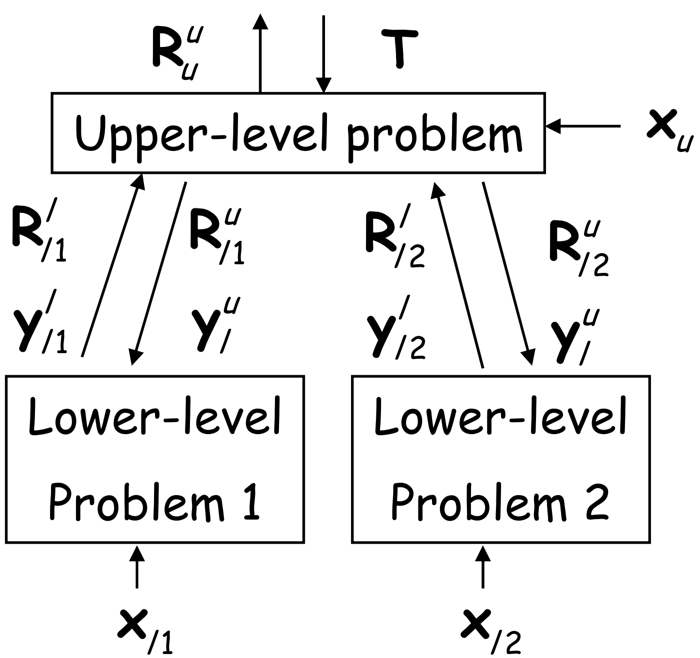
$\mathbf{g}_u(\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \mathbf{x}_u) \le \mathbf{0}$

$\mathbf{h}_u(\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \mathbf{x}_u) = \mathbf{0}$

where $\mathbf{R}_u^u = \mathbf{r}_u(\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \mathbf{x}_u)$

Two subsystems, with shared variables

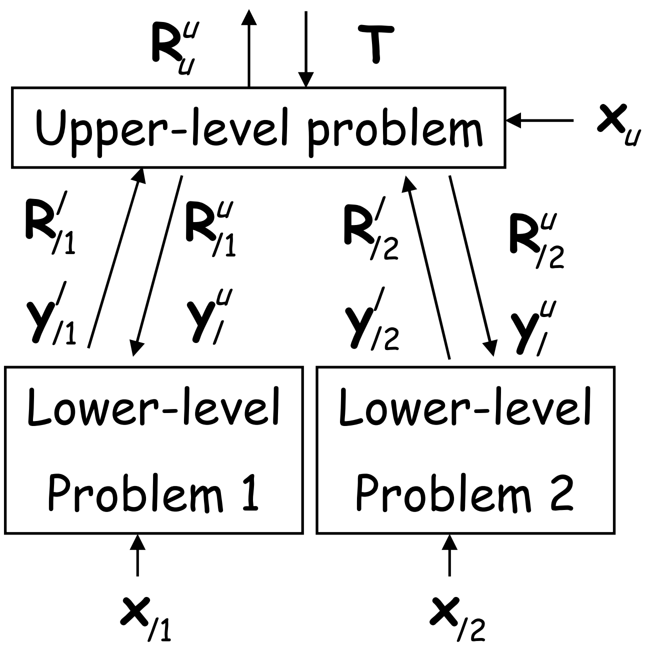
Upper-level problem formulation, first ATC iteration



$$
\begin{aligned}
\text{minimize} \quad & \left\| \mathbf{R}_u^u - \mathbf{T} \right\|_2^2 \\
\text{with respect to} \quad & \mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u, \mathbf{y}_l^u \\
\text{subject to} \quad & \mathbf{g}_u(\mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u) \leq \mathbf{0} \\
& \mathbf{h}_u(\mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u) = \mathbf{0} \\
\text{where} \quad & \mathbf{R}_u^u = \mathbf{r}_u(\mathbf{R}_{/1}^u, \mathbf{R}_{/2}^u, \mathbf{x}_u)
\end{aligned}
$$

Two subsystems, with shared variables

Lower-level problem formulation, ATC iterations 1,2,…,n



minimize $\left\|\mathbf{R}_{lk}^l - \mathbf{R}_{lk}^u\right\|_2^2 + \left\|\mathbf{y}_{lk}^l - \mathbf{y}_l^u\right\|_2^2$

with respect to $\mathbf{x}_{lk}, \mathbf{y}_{lk}^l$

subject to $\mathbf{g}_{lk}(\mathbf{x}_{lk}, \mathbf{y}_{lk}^l) \leq \mathbf{0}$

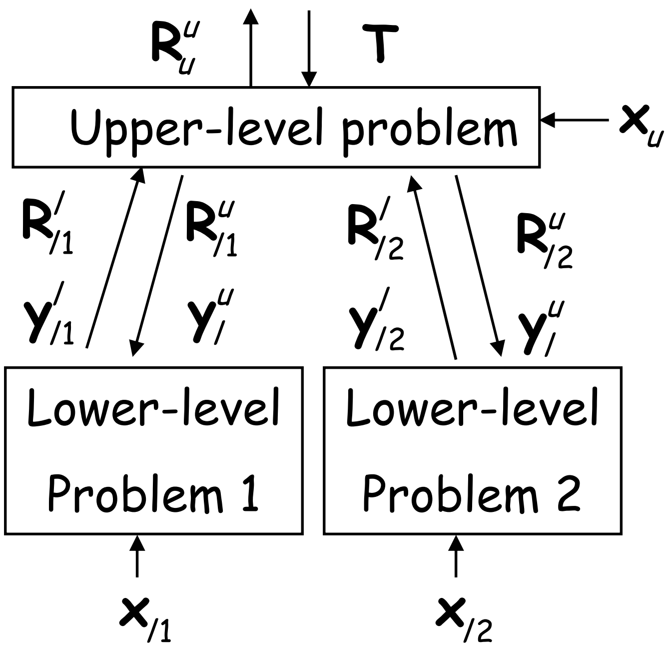
$\mathbf{h}_{lk}(\mathbf{x}_{lk}, \mathbf{y}_{lk}^l) = \mathbf{0}$

where $\mathbf{R}_{lk}^l = \mathbf{r}_{lk}(\mathbf{x}_{lk}, \mathbf{y}_{lk}^l)$

for k=1,2

Two subsystems, with shared variables

Upper-level problem formulation, ATC iterations 2,3,…,n



minimize $\left\| \mathbf{R}^u_u - \mathbf{T} \right\|_2^2 + \varepsilon^u_R + \varepsilon^u_y$

with respect to $\mathbf{R}^u_{/1}, \mathbf{R}^u_{/2}, \mathbf{x}_u, \mathbf{y}^u_l, \varepsilon^u_R, \varepsilon^u_y$

subject to
$$\left\| \mathbf{R}^u_{/1} - \mathbf{R}^l_{/1} \right\|_2^2 + \left\| \mathbf{R}^u_{/2} - \mathbf{R}^l_{/2} \right\|_2^2 \leq \varepsilon^u_R$$
$$\left\| \mathbf{y}^u_l - \mathbf{y}^l_{/1} \right\|_2^2 + \left\| \mathbf{y}^u_l - \mathbf{y}^l_{/2} \right\|_2^2 \leq \varepsilon^u_y$$
$$\mathbf{g}_u(\mathbf{R}^u_{/1}, \mathbf{R}^u_{/2}, \mathbf{x}_u) \leq 0$$
$$\mathbf{h}_u(\mathbf{R}^u_{/1}, \mathbf{R}^u_{/2}, \mathbf{x}_u) = 0$$

where $\mathbf{R}^u_u = \mathbf{r}_u(\mathbf{R}^u_{/1}, \mathbf{R}^u_{/2}, \mathbf{x}_u)$

General Bi-level Case

Upper-level problem formulation

minimize $\left\| \mathbf{R}_u^u - \mathbf{T} \right\|_2^2 + \varepsilon_R^u + \varepsilon_y^u$

with respect to $\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \ldots, \mathbf{R}_{lm}^u, \mathbf{x}_u, \mathbf{y}_l^u, \varepsilon_R^u, \varepsilon_y^u$

subject to $\sum_{k=1}^{m} \left\| \mathbf{R}_{lk}^u - \mathbf{R}_{lk}^l \right\|_2^2 \leq \varepsilon_R^u$

$\sum_{k=1}^{m} \left\| \mathbf{S}_k \mathbf{y}_l^u - \mathbf{y}_{lk}^l \right\|_2^2 \leq \varepsilon_y^u$

$\mathbf{g}_u(\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \ldots, \mathbf{R}_{lm}^u, \mathbf{x}_u) \leq \mathbf{0}$

$\mathbf{h}_u(\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \ldots, \mathbf{R}_{lm}^u, \mathbf{x}_u) = \mathbf{0}$

where $\mathbf{R}_u^u = \mathbf{r}_u(\mathbf{R}_{l1}^u, \mathbf{R}_{l2}^u, \ldots, \mathbf{R}_{lm}^u, \mathbf{x}_u)$

terms in red appear in ATC iterations 2 to n

General Bi-level Case

Lower-level problem formulation

minimize $\left\| \mathbf{R}_{lk}^{l} - \mathbf{R}_{lk}^{u} \right\|_2^2 + \left\| \mathbf{y}_{lk}^{l} - \mathbf{S}_k \mathbf{y}_l^{u} \right\|_2^2$

with respect to $\mathbf{x}_k, \mathbf{y}_{lk}^{l}$

subject to $\mathbf{g}_{lk}(\mathbf{x}_{lk}, \mathbf{y}_{lk}^{l}) \le \mathbf{0}$

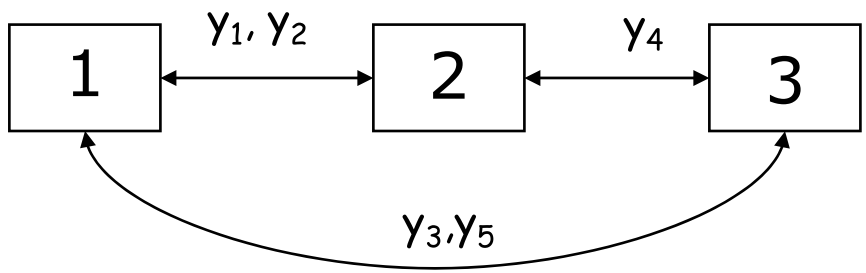
$\mathbf{h}_{lk}(\mathbf{x}_{lk}, \mathbf{y}_{lk}^{l}) = \mathbf{0}$

where $\mathbf{R}_{lk}^{l} = \mathbf{r}_{lk}(\mathbf{x}_k, \mathbf{y}_{lk}^{l})$

for each supbroblem k

Note: S_k is a so-called selection matrix

Selection Matrix Example



vector of all shared variables to be coordinated by parent: $\mathbf{y}_l^u = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$

selection matrix for child 1: $\mathbf{S}_1 = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{bmatrix}$; vector shared variables for child 1: $\mathbf{y}_{l1}^l \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_5 \end{bmatrix}$

selection matrix for child 2: $\mathbf{S}_2 = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \end{bmatrix}$; vector shared variables for child 2: $\mathbf{y}_{l2}^l \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_4 \end{bmatrix}$

ATC Process Termination

- Typically, the ATC process is iterated until the consistency constraint optimization variables (the epsilon terms) cannot be reduced anymore
- A useful additional criterion is the change of the values of the optimization variables before and after an ATC iteration for each subproblem. If they don't change significantly anymore for all the subproblems, the ATC process may be terminated.
- It is possible that the ATC process will not decrease the epsilon terms under desired thresholds or will not satisfy design targets; in this case one has to reconsider target values and/or the feasible design space

Example

$$\min_{z\geq 0} \quad z_1^2 + z_2^2$$

subject to

$$z_3^{-2} + z_4^2 - z_5^2 \leq 0$$
$$z_5^2 + z_6^{-2} - z_7^2 \leq 0$$
$$z_8^2 + z_9^2 - z_{11}^2 \leq 0$$
$$z_8^{-2} + z_{10}^2 - z_{11}^2 \leq 0$$
$$z_{11}^2 + z_{12}^{-2} - z_{13}^2 \leq 0$$
$$z_{11}^2 + z_{12}^2 - z_{14}^2 \leq 0$$
$$z_1^2 - z_3^2 - z_4^{-2} - z_5^2 = 0$$
$$z_2^2 - z_5^2 - z_6^2 - z_7^2 = 0$$
$$z_3^2 - z_8^2 - z_9^{-2} - z_{10}^{-2} - z_{11}^2 = 0$$
$$z_6^2 - z_{11}^2 - z_{12}^2 - z_{13}^2 - z_{14}^2 = 0$$

We use the equality constraints to decompose this problem into two subproblems; thus we have a bi-level hierarchy with two subsystems and one shared variable

Upper-level Problem

$$\text{minimize} \quad \left\|\mathbf{R}_u^u - \mathbf{T}\right\|_2^2 + \varepsilon_R^u + \varepsilon_y^u$$

$$\text{with respect to} \quad R_{l1}^u, R_{l2}^u, x_{u,1}, x_{u,2}, x_{u,3}, y_l^u, \varepsilon_R^u, \varepsilon_y^u$$

$$\text{subject to} \quad (R_{l1}^u - R_{l1}^l)^2 + (R_{l2}^u - R_{l2}^l)^2 \le \varepsilon_R^u$$

$$(y_l^u - y_{l1}^u)^2 + (y_l^u - y_{l2}^u)^2 \le \varepsilon_y^u$$

$$(R_{l1}^u)^{-2} + (x_{u,1})^2 - (x_{u,2})^2 \le 0$$

$$(x_{u,2})^2 + (R_{l2}^u)^{-2} - (x_{u,3})^2 \le 0$$

$$\text{where} \quad \mathbf{R}_u^u = \begin{bmatrix} \sqrt{(R_{l1}^u)^2 + (x_{u,1})^{-2} + (x_{u,2})^2} \\ \sqrt{(x_{u,2})^2 + (R_{l2}^u)^2 + (x_{u,3})^2} \end{bmatrix}; \quad \mathrm{T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{with} \quad R_{u,1}^u := z_1 \;, R_{u,1}^u := z_2 \;, \; R_{l1}^u := z_3 \;, \; R_{l2}^u := z_6$$

$$x_{u,1} := z_4 \;, \; x_{u,2} := z_5 \;, \; x_{u,3} := z_7 \;, \; y_l^u := z_{11}$$

Lower-level Problem 1

minimize	$(R_{/1}^{l} - R_{/1}^{u})^2 + (y_{/1}^{l} - y_{/}^{u})^2$
with respect to	$x_{/1,1}, x_{/1,2}, x_{/1,3}, y_{/1}^{l}$
subject to	$(x_{/1,1})^2 + (x_{/1,2})^2 - (y_{/1}^{l})^2 \leq 0$
	$(x_{/1,1})^{-2} + (x_{/1,3})^2 - (y_{/1}^{l})^2 \leq 0$
where	$R_{/1}^{l} = \sqrt{(x_{/1,1})^2 + (x_{/1,2})^{-2} + (x_{/1,3})^{-2} + (y_{/1}^{l})^2}$
with	$x_{/1,1} := z_8$, $x_{/1,2} := z_9$, $x_{/1,3} := z_{10}$, $y_{/1}^{l} := z_{11}$

Lower-level Problem 2

minimize $(R^l_{l2} - R^u_{l2})^2 + (y^l_{l2} - y^u_l)^2$

with respect to $x_{l2,1}, x_{l2,2}, x_{l2,3}, y^l_{l2}$

subject to $(y^l_{l2})^2 + (x_{l2,1})^{-2} - (x_{l2,2})^2 \leq 0$

$(y^l_{l2})^2 + (x_{l2,2})^2 - (x_{l2,3})^2 \leq 0$

where $R^l_{l2} = \sqrt{(y^l_{l2})^2 + (x_{l2,1})^2 + (x_{l2,2})^2 + (x_{l2,3})^2}$

with $x_{l2,1} := z_{12}$, $x_{l2,2} := z_{13}$, $x_{l2,3} := z_{14}$, $y^l_{l2} := z_{11}$